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Interlayer Shear Slip Theory for Cross-Ply Laminates with Nonrigid Interfaces

X. Lu* and D. Liu†
Michigan State University, East Lansing, Michigan 48824

In the conventional analysis for laminated composite materials, the composite interface is always assumed to be rigidly bonded. However, due to the low shear modulus and poor bonding, the composite interface can be nonrigid. Based on the previously developed interlaminar shear stress continuity theory and a linear shear slip law, this study presents a so-called interlayer shear slip theory to investigate the effect of the interfacial bonding on the behavior of cross-ply laminates. Closed-form solutions for the cases of the cylindrical bending of long composite strips with [0/0] and [0/90] sequences are obtained. Numerical results for the laminates with different length-to-thickness ratios are presented. They reveal that the interlayer shear slip theory is valid for cross-ply laminates with rigid and nonrigid interfaces. It is also concluded from this study that at some special locations, namely, singular points, the transverse shear stress or in-plane normal stress remains insensitive to the condition of interfacial bonding.

Introduction

FIBER-REINFORCED polymer-matrix composite materials have high in-plane strength but low density. They are excellent materials for high-performance structures. In studying composite structures, classical laminate theory (CLT) has been widely used for stress analysis. However, CLT is only accurate for thin composite laminates.[1,2] For thick composite laminates, the transverse shear deformation should be considered. In addition to thickness, the low shear modulus of polymer matrices also has a significant effect on the transverse shear deformation.[3,4] Consequently, the transverse shear deformation is an important issue in composite analysis.

Many techniques have been developed for composite stress analysis. Comprehensive reviews of these techniques can be found in several articles.[5-7] Among the different techniques reported, the one that has received the most attention in recent years is the so-called high-order shear deformation theory (HSDT). By introducing high-order terms for in-plane displacements, many high-order theories[7-12] have been presented for both displacement and in-plane stress analysis. Based on a layer-wise displacement field, Reddy[13] has indicated that the high-order theories can be summed up by a so-called generalized laminated plate theory (GLPT).

In general, GLPT can give accurate results for displacements and in-plane stresses in laminate analysis. However, it may not be suitable for interlaminar stress. In composite analysis, due to the high interlaminar stresses and weak bonding between the composite layers, delamination can occur easily on the composite interface. Two types of delamination, namely, edge delamination[14] and central delamination,[15] have been widely investigated. Both of them can be viewed as a result of interlaminar stress concentration caused by material property mismatch in the thickness direction. Because delamination can greatly reduce the integrity of composite structures, the study of interlaminar stresses has become an important issue in composite analysis. Since GLPT does not take the interlaminar stress continuity conditions into consideration, the interlaminar stresses cannot be obtained from the constitutive equations directly. Although it is possible to recover the interlaminar stresses through the equilibrium equations,[16] it is tedious and not suitable for structures with complex configurations.

To include the continuity of interlaminar stresses on the composite interface, it is necessary to formulate the composite laminate layer by layer. Ambartsumyan[17] was among the earliest to present a multiple-layer technique in the composite laminate analysis. Based on a parabolic distribution for the transverse shear stresses in the composite layers, he presented a shear deformation theory that satisfied the continuity conditions. Similar to this work, a refined theory for multilayered symmetric plates was presented by Librescu and Reddy.[18] Another stress-based technique that included the interlaminar stress continuity was presented by Mau et al.[19] This technique was named the hybrid-stress finite element method. Spilker[20] and many other investigators extended this technique for studies with high-order stress assumptions. In addition to the finite element analysis, Pagano[21] assumed a stress distribution in each layer. He derived the governing equations for laminate analysis with use of a variational approach. The continuity of interlaminar stresses was also satisfied in his formulation.

Instead of assuming stresses, DiSciuva[22] presented a displacement-based approach that had piecewise linear in-plane displacements through the thickness while the out-of-plane displacement was constant. A variational method was used to formulate the governing equations. However, due to the low order of the assumed displacement field, the transverse shear stresses were constant through the thickness. Toledano and Murakami[23] used a similar displacement field in their analysis. However, they also assumed quadratic transverse shear stress distributions across each individual layer independently. Reissner's mixed variational principle[24] was used together with the interlaminar shear stress continuity conditions in their formulation. This technique was valid for improving in-plane deformation. However, the interlaminar shear stresses were suggested to be recovered by equilibrium equations because of accuracy reasons. In addition, by imposing a cubic spline function to model the through-the-thickness deformation, Hinrichsen and Palazotto[25] presented a quasi-three-dimensional nonlinear finite element analysis for thick composite plates. This theory was successfully applied to study the transverse deflection and in-plane stresses and deformation in composite laminates with various thicknesses.

In view of the advantages and disadvantages of the techniques reported, it was concluded that an accurate theory for interlaminar stress analysis should consider the transverse shear

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*Graduate Student, Department of Metallurgy, Mechanics, and Material Science.

†Associate Professor, Department of Metallurgy, Mechanics, and Material Science. Member AIAA.

effect and the continuity requirements for both displacements and interlaminar stresses on the composite interface. The interlaminar stresses can then be obtained directly from the constitutive equations instead of the equilibrium equations. Besides, it is important that the formulation should be variational consistent[12] and can be extended to finite element analysis for structures with complex configurations. Based on these understandings, an interlaminar shear stress continuity theory (ISSCT) was develeoped by the authors[26,27] with a displacement field refined from the GLPT.[13] Because of its quasi-three-dimension in nature, this technique was suitable for both thick and thin composite laminates.

Nonrigidly Bonded Interface

All of the theories just reviewed are based on the assumption of rigid bonding, i.e., the displacement on the composite interface remains to be single value. It has been recognized that the interfacial condition can strongly affect the structure performance. Since the composite interface is mainly dominated by the matrix of low shear modulus, the interlayer connection may not be rigid. Therefore, in order to accurately predict the composite performance, it is necessary to account for the bonding effect in composite analysis.

Pioneering work in the study of nonrigidly bonded interface in composite structures was performed by Newmark et al.[28] Based on the Bernoulli-Euler beam theory, a laminated beam theory with a shear slip on the layer interface was developed. A linear slip law was employed to model the nonrigid connection between the layers. One of the applications of their theory was for the analysis of two beams connected by nails. The theory was later extended for plate analysis by subsequent investigators. In addition, different slip laws were also introduced for nonrigid interface analysis.[29-32] However, due to the fundamental assumptions of the beam theory and the plate theory, the transverse shear effect was not considered in these studies. To the authors' best knowledge, Toledano and Murakami[33] were the first to use a laminate theory, which accounts for both transverse shear effect and interlaminar shear stress continuity, to study nonrigid bonding effect. They presented a technique that incorporated a slip law in their laminate theory.[23] However, the interlaminar shear stresses that governed the shear slip at the interface were not accurately predicted by their theory due to the low order of the assumed displacements and stresses. Instead of laminated analysis, elasticity studies on sandwich beams with nonrigid bonding were presented by Rao and Ghosh[34] and Fazio et al.[35] Because of the analytical complexity, only limited cases were examined.

The objective of this study was to present a technique for laminated composites with various types of bonding on composite interface. As mentioned earlier, both transverse shear effect and interlaminar stress continuity condition played crucial roles in composite stress analysis. It then was the objective of this study to extend the ISSCT for composite laminates with nonrigidly bonded interface. A so-called interlayer shear slip theory (ISST) based on a linear shear slip law was developed. Composite laminates with stacking sequences of [0/0] and [90/0] under cylindrical bending were investigated. In order to verify the ISST, an embedded layer approach was also used to study the same problems. Results from both studies were compared to assess the new theory.

Interlayer Shear Slip Theory

Displacement Field

A composite laminate composed of n layers, as shown in Fig. 1, is considered. A Cartesian coordinate system is chosen such that the middle plane of the laminate occupies a domain Ω in the x-y plane while the z axis is normal to this plane. The displacement field at a generic point (x, y, z) in the laminate is assumed to be as follows:

$$u(x,y,z) = \sum_{i=1}^{n} \{U_{i-1}^{(T)}(x,y)\phi_1^{(i)}(z) + U_i^{(B)}(x,y)\phi_2^{(i)}(z) + S_{2i-1}(x,y)\phi_3^{(i)}(z) + S_{2i-1}(x,y)\phi_4^{(i)}(z)\} \tag{1a}$$

$$v(x,y,z) = \sum_{i=1}^{n} \{V_{i-1}^{(T)}(x,y)\phi_1^{(i)}(z) + V_i^{(B)}(x,y)\phi_2^{(i)}(z) + T_{2i-1}(x,y)\phi_3^{(i)}(z) + T_{2i-1}(x,y)\phi_4^{(i)}(z)\} \tag{1b}$$

$$w(x,y,z) = w(x,y) \tag{1c}$$

where $\phi_j^{(i)}$ are Hermite cubic shape functions and can be expressed as

$$\begin{aligned}
\phi_1^{(i)} &= 1 - 3[(z - z_{i-1})/h_i]^2 + 2[(z - z_{i-1})/h_i]^3 \\
\phi_2^{(i)} &= 3[(z - z_{i-1})/h_i]^2 - 2[(z - z_{i-1})/h_i]^3 \qquad z_{i-1} \le z \le z_i \\
\phi_3^{(i)} &= (z - z_{i-1})[1 - (z - z_{i-1})/h_i]^2 \\
\phi_4^{(i)} &= (z - z_{i-1})^2\{[(z - z_{i-1})/h_i - 1]/h_i\} \\
\phi_1^{(i)} &= \phi_2^{(i)} = \phi_3^{(i)} = \phi_4^{(i)} = 0 \qquad z < z_{i-1} \quad \text{or} \quad z > z_i
\end{aligned} \tag{2}$$

The superscript (i) represents layer number, i.e., the ith layer of the composite laminate, and h_i the thickness of the layer. As shown in Fig. 1, $U_i^{(B)}$ and $V_i^{(B)}$ are the displacements at a point (x, y, z_i) in layer (i), and $U_i^{(T)}$ and $V_i^{(T)}$ are at the same point but in layer $(i + 1)$. Also shown in Fig. 1, S and T are the first derivatives of u and v with respect to the z axis, respectively. More specifically, S_{2i} and T_{2i} represent the nodal values of $\partial u/\partial z$ and $\partial v/\partial z$ at a point (x, y, z_i) in layer $(i + 1)$ whereas S_{2i-1} and T_{2i-1} are at the same point but in layer (i). The displacement w in the thickness direction is assumed to be constant due to the relatively small value of transverse normal stress σ_z compared to other stresses.[17,36] Accordingly, the total number of assumed variables is $8n + 1$. However, it is noted that the importance of σ_z is debatable under some circumstance.[37] A study about the effect of σ_z in perfectly bonded laminates can be found in Ref. 38.

If the composite laminate of interest is of cross-ply sequence, the constitutive equations for individual layer can be expressed by the following equations.[36]

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{bmatrix}^{(i)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 \\ 0 & 0 & 0 & 2Q_{66} \end{bmatrix}^{(i)} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_{xy} \end{bmatrix}^{(i)} \tag{3}$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix}^{(i)} = \begin{bmatrix} 2Q_{44} & 0 \\ 0 & 2Q_{55} \end{bmatrix}^{(i)} \begin{bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{bmatrix}^{(i)}$$

For linear strain-displacement consideration, the following equations can be employed.

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\right)$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{4}$$

In addition, on the composite interface, the displacements and interlaminar shear stresses will have the following relations, i.e.,

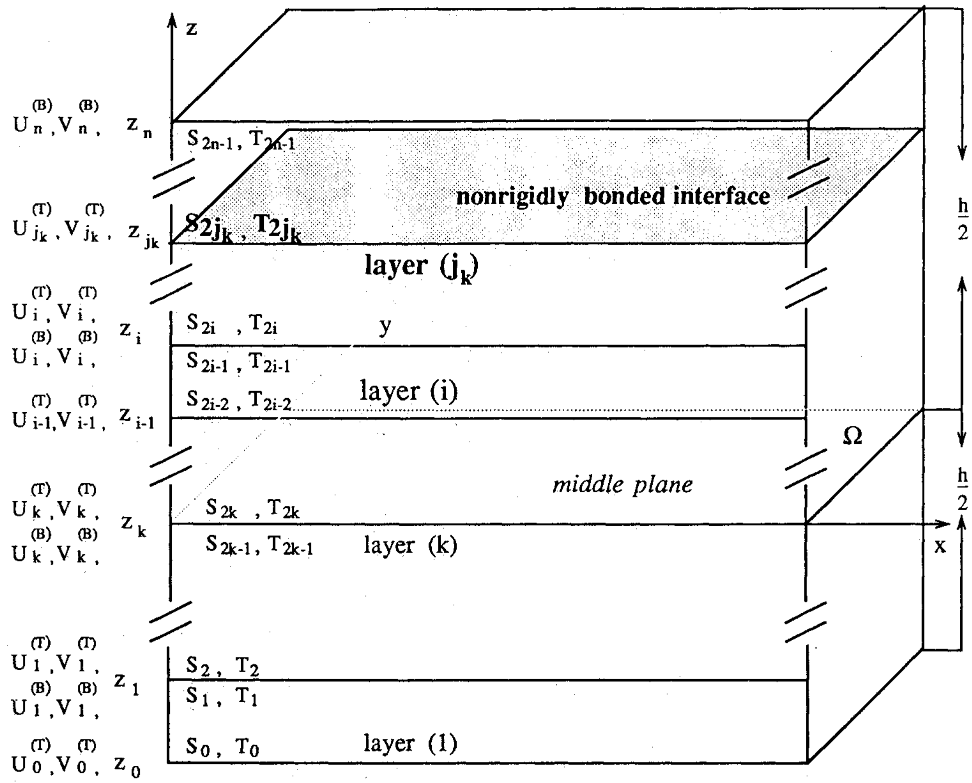


Fig. 1 Nodal variables and the coordinate system.

$$\Delta U_i = U_i^{(T)} - U_i^{(B)}, \qquad \Delta V_i = V_i^{(T)} - V_i^{(B)}$$

$$\lim_{z\to z_i} \tau_{xz}^{(i+1)} = \lim_{z\to z_i} \tau_{xz}^{(i)}, \qquad \lim_{z\to z_i} \tau_{yz}^{(i+1)} = \lim_{z\to z_i} \tau_{yz}^{(i)}$$

$$i = 1, 2, \ldots, n-1 \tag{5}$$

where ΔU_i and ΔV_i are interlayer shear slips.

Rigid Interface

If the composite layers are rigidly bonded, there will be no slip on the interfaces, i.e., $\Delta U_i = \Delta V_i = 0$, where $i = 1, 2, \ldots, n-1$. These conditions imply that

$$U_i^{(T)} = U_i^{(B)} = U_i, \qquad V_i^{(T)} = V_i^{(B)} = V_i \tag{6}$$

By substituting Eqs. (1) into Eqs. (4), then combining with Eqs. (3), the stresses can be expressed in terms of displacements. With use of the continuity conditions for interlaminar shear stresses in Eqs. (5), S_{2i-1} and T_{2i-1} can be verified to be functions of S_{2i} and T_{2i}, respectively,

$$S_{2i-1} = \frac{Q_{55}^{(i+1)}}{Q_{55}^{(i)}} S_{2i} + \left[\frac{Q_{55}^{(i+1)}}{Q_{55}^{(i)}} - 1\right] \frac{\partial w}{\partial x}$$
$$T_{2i-1} = \frac{Q_{44}^{(i+1)}}{Q_{44}^{(i)}} T_{2i} + \left[\frac{Q_{44}^{(i+1)}}{Q_{44}^{(i)}} - 1\right] \frac{\partial w}{\partial y} \tag{7}$$

If the top and bottom surfaces are traction free, the following equations are valid, i.e.,

$$\tau_{xz}|_{z=\pm h/2} = 0, \qquad \tau_{yz}|_{z=\pm h/2} = 0 \tag{8}$$

where h is the total thickness of the composite laminate. With the same fashion as used in obtaining Eqs. (7), four variables can be further eliminated:

$$S_0 = S_{2n-1} = -\frac{\partial w}{\partial x}, \qquad T_0 = T_{2n-1} = -\frac{\partial w}{\partial y} \tag{9}$$

It then is concluded that it needs only four variables, U_i, V_i, S_{2i}, and T_{2i}, for each nodal point. Consequently, the total number of the independent variables is reduced to $4n + 1$. The reduced variables are assigned new notations and shown in Fig. 2. The displacement field can then be rewritten as follows,

$$u(x, y, z) = \sum_{j=0}^{n} U_j \Phi^j + \sum_{j=1}^{n-1} \bar{S}_j \Psi_1^j + \left[\sum_{j=1}^{n-1} \left(\frac{Q_{55}^{(j+1)}}{Q_{55}^{(j)}} - 1\right) \Theta_j^{(1)} - \phi_3^{(1)} - \phi_4^{(n)}\right] \frac{\partial w}{\partial x} \tag{10a}$$

$$v(x, y, z) = \sum_{j=0}^{n} V_j \Phi^j + \sum_{j=1}^{n-1} \bar{T}_j \Psi_2^j + \left[\sum_{j=1}^{n-1} \left(\frac{Q_{44}^{(j+1)}}{Q_{44}^{(j)}} - 1\right) \Theta_j^{(1)} - \phi_3^{(1)} - \phi_4^{(n)}\right] \frac{\partial w}{\partial y} \tag{10b}$$

$$w(x, y, z) = w(x, y) \tag{10c}$$

The shape functions are given by the following equations:

$$\Phi^j = \begin{cases} \phi_1^{(i)} & j = i-1 \\ \phi_2^{(i)} & j = i \end{cases} \quad \text{layer } (i) \tag{11a}$$

$$\Phi^j = 0 \qquad \text{others} \tag{11b}$$

$$\Psi_1^j = \begin{cases} \phi_3^{(i)} & j = i-1 \\ \dfrac{Q_{55}^{(i+1)}}{Q_{55}^{(i)}} (\phi_4^{(i)}) & j = i \end{cases} \quad \text{layer } (i) \tag{11c}$$

$$\Psi_1^j = 0 \qquad \text{others} \tag{11d}$$

$$\Psi_2^j = \begin{cases} \phi_3^{(i)} & j = i-1 \\ \dfrac{Q_{44}^{(i+1)}}{Q_{44}^{(i)}} (\phi_4^{(i)}) & j = i \end{cases} \quad \text{layer } (i) \tag{11e}$$

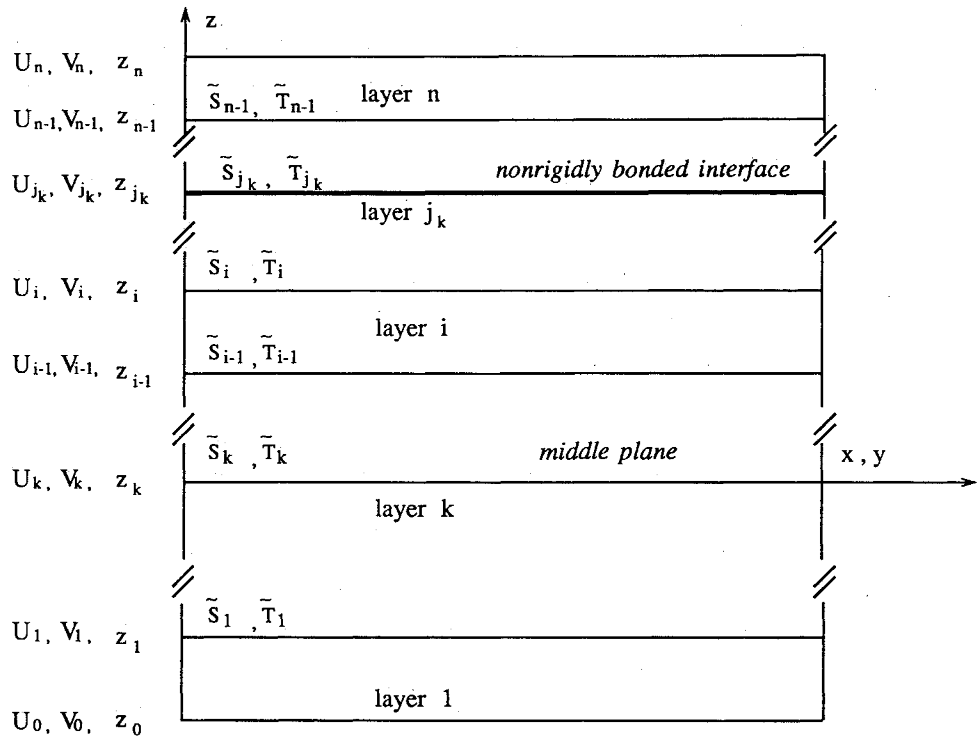


Fig. 2 Reduced variables.

$$\Psi_2^j = 0 \qquad \text{others} \tag{11f}$$

$$\Theta_j^{(1)} = \begin{cases} \phi_4^{(i)} & j = i \quad \text{layer } (i) \\ 0 & \text{others} \end{cases} \tag{11g}$$

Nonrigid Interface

Assume the composite laminate of interest has $n_1 (n_1 \leq n - 1)$ nonrigidly bonded interfaces that are located between layer (j_k) and layer $(j_k + 1)$, as shown in Fig. 1. A set Π that contains all nonrigid interface j_k, $k = 1, 2, \ldots,$ and n_1, is defined. In this study, a linear shear slip law presented by Newmark et al.[28] is employed for nonrigid bonding, i.e.,

$$\begin{aligned} \Delta U_{j_k} &= \mu_{j_k} \tau_{xz}^{(j_k)}, \qquad \mu_{j_k} \geq 0 \\ \Delta V_{j_k} &= \nu_{j_k} \tau_{yz}^{(j_k)}, \qquad \nu_{j_k} \geq 0 \end{aligned} \tag{12}$$

where $j_k \in \Pi$. The coefficients μ and ν are used to represent the interfacial shear bonding condition. For rigid bonding, μ and ν vanish. On the other hand, μ and ν go to $+\infty$ if there is no bonding on the interface, e.g., the interface is delaminated. Since the shear slips ΔU and ΔV should be finite values, τ_{xz} and τ_{yz} will vanish on the delaminated interface.

Substituting Eqs. (12) into Eqs. (5), the displacements above and below the nonrigid interface will have the following relations:

$$\begin{aligned} U_{j_k}^{(B)} &= U_{j_k} - \mu_{j_k} \tau_{xz}^{(j_k)}, \qquad U_{j_k} = U_{j_k}^{(T)} \\ V_{j_k}^{(B)} &= V_{j_k} - \nu_{j_k} \tau_{yz}^{(j_k)}, \qquad V_{j_k} = V_{j_k}^{(T)} \end{aligned} \tag{13}$$

where $j_k \in \Pi$. Since the continuity of interlaminar shear stresses is also true for nonrigid interface, Eqs. (7) remain valid. Therefore, it still requires four variables, U_i, V_i, S_{2i}, and T_{2i}, to express each nodal point. Thus, the total number of the independent variables remains to be $4n + 1$. The displacement field for a composite laminate with n_1 nonrigidly bonded interfaces then becomes

$$u = \sum_{j=0}^{n} U_j \Phi^j + \sum_{j=1}^{n-1} \tilde{S}_j (\Psi_1^j - \mu_j Q_{55}^{(j+1)} \Theta_j^{(2)}) + \left\{ \sum_{j=1}^{n-1} \left[\left(\frac{Q_{55}^{(j+1)}}{Q_{55}^{(i)}} - 1 \right) \Theta_j^{(1)} - \mu_j Q_{55}^{(j+1)} \Theta_j^{(2)} \right] - \phi_3^{(1)} - \phi_4^{(n)} \right\} \frac{\partial w}{\partial x} \tag{14a}$$

$$v = \sum_{j=0}^{n} V_j \Phi^j + \sum_{j=1}^{n-1} \tilde{T}_j (\Psi_2^j - \nu_j Q_{44}^{(j+1)} \Theta_j^{(2)}) + \left\{ \sum_{j=1}^{n-1} \left[\left(\frac{Q_{44}^{(j+1)}}{Q_{44}^{(i)}} - 1 \right) \Theta_j^{(1)} - \nu_j Q_{44}^{(j+1)} \Theta_j^{(2)} \right] - \phi_3^{(1)} - \phi_4^{(n)} \right\} \frac{\partial w}{\partial y} \tag{14b}$$

$$w = w(x, y) \tag{14c}$$

where

$$\mu_j = \nu_j = 0 \qquad \text{if} \quad j \notin \Pi$$
$$\Theta_j^{(2)} = \begin{cases} \phi_2^{(i)} & j = i \quad \text{layer } (i) \\ 0 & \text{others} \end{cases} \tag{15}$$

Equilibrium Equations

The principle of virtual displacement is used to derive the equilibrium equations and associated boundary conditions. The principle of virtual displacement in this study can be stated as

$$\int_{-h/2}^{h/2} \int_{\Omega} (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + 2\tau_{xy} \delta\varepsilon_{xy} + 2\tau_{xz} \delta\varepsilon_{xz} + 2\tau_{yz} \delta\varepsilon_{yz}) \, dA \, dz + \sum_{j=1}^{n} \int_{\Omega} (\tau_{xz}^{(j)} \delta\Delta U_j + \tau_{yz}^{(j)} \delta\Delta V_j) \, dA - \int_{\Omega} P_z \delta w \, dA = 0 \tag{16}$$

Assume that μ and ν are independent of x and y. In other words, the bonding condition on each interface is uniform. By substituting Eqs. (14) into Eqs. (4), the strains can then be expressed in terms of displacement variables as

$$\varepsilon_x = \sum_{j=0}^{n} U_{j,x} \Phi^j + \sum_{j=1}^{n-1} \tilde{S}_{j,x} \Psi_1^j + \lambda_1 \frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \sum_{j=0}^{n} V_{j,y}\Phi^j + \sum_{j=1}^{n-1} \tilde{T}_{j,y}\bar{\Psi}_2^j + \lambda_2 \frac{\partial^2 w}{\partial y^2}$$

$$\varepsilon_{xz} = \frac{1}{2}\left\{\sum_{j=0}^{n} U_j\Phi^j_{,z} + \sum_{j=1}^{n-1} \tilde{S}_j\bar{\Psi}^j_{1,z} + \left(\frac{d\lambda_1}{dz}+1\right)\frac{\partial w}{\partial x}\right\}$$

$$\varepsilon_{yz} = \frac{1}{2}\left\{\sum_{j=0}^{n} V_j\Phi^j_{,z} + \sum_{j=1}^{n-1} \tilde{T}_j\bar{\Psi}^j_{2,z} + \left(\frac{d\lambda_2}{dz}+1\right)\frac{\partial w}{\partial y}\right\}$$

$$\varepsilon_{xy} = \frac{1}{2}\left\{\sum_{j=0}^{n} (U_{j,y}+V_{j,x})\Phi^j + \sum_{j=1}^{n-1} (\tilde{S}_{j,y}\bar{\Psi}_1^j + \tilde{T}_{j,x}\bar{\Psi}_2^j) + (\lambda_1+\lambda_2)\frac{\partial^2 w}{\partial x \partial y}\right\} \tag{17}$$

where

$$\lambda_1 = \left\{\sum_{j=1}^{n-1}\left[\left(\frac{Q_{55}^{(j+1)}}{Q_{55}^{(i)}} - 1\right)\Theta_j^{(1)} - \mu_j Q_{55}^{(j+1)}\Theta_j^{(2)}\right] - \phi_3^{(1)} - \phi_4^{(n)}\right\}$$

$$\lambda_2 = \left\{\sum_{j=1}^{n-1}\left[\left(\frac{Q_{44}^{(j+1)}}{Q_{44}^{(i)}} - 1\right)\Theta_j^{(1)} - \nu_j Q_{44}^{(j+1)}\Theta_j^{(2)}\right] - \phi_3^{(1)} - \phi_4^{(n)}\right\}$$

$$\bar{\Psi}_1^j = (\Psi_1^j - \mu_j Q_{55}^{(j+1)}\Theta_j^{(2)})$$

$$\bar{\Psi}_2^j = (\Psi_2^j - \nu_j Q_{44}^{(j+1)}\Theta_j^{(2)}) \tag{18}$$

By substituting Eqs. (17) along with Eqs. (3) into Eq. (16), after integrating by parts and collecting similar terms, the governing equations become

δw:

$$Q_{x,x} + Q_{y,y} - \bar{\lambda}_{x,xx} - \bar{\lambda}_{xy,xy} - \bar{\lambda}_{y,yy} + \bar{\eta}_{x,x} + \bar{\eta}_{y,y} + P_z + \sum_{k=1}^{n_1}\left[\mu_{jk}(Q_{55}^{jk+1})^2\left(\tilde{S}_{jk,x} + \frac{\partial^2 w}{\partial x^2}\right) + \nu_{jk}(Q_{44}^{jk+1})^2\left(\tilde{T}_{jk,y} + \frac{\partial^2 w}{\partial y^2}\right)\right] = 0$$

δU_j:

$$N^j_{x,x} + N^j_{xy,y} - Q^j_x = 0 \qquad j = 0, 1, \ldots, n$$

δV_j:

$$N^j_{xy,x} + N^j_{y,y} - Q^j_y = 0 \qquad j = 0, 1, \ldots, n$$

$\delta\tilde{S}_j$:

$$M^j_{x,x} + M^j_{1xy,y} - R^j_x - \mu_j(Q_{55}^{j+1})^2\left(\tilde{S}_j + \frac{\partial w}{\partial x}\right)\delta_{jj_k} = 0$$

$$j = 1, 2, \ldots, n-1 \quad j_k \in \Pi$$

$\delta\tilde{T}_j$:

$$M^j_{2xy,x} + M^j_{y,y} - R^j_y - \nu_j(Q_{44}^{j+1})^2\left(\tilde{T}_j + \frac{\partial w}{\partial y}\right)\delta_{jj_k} = 0$$

$$j = 1, 2, \ldots, n-1 \quad j_k \in \Pi \tag{19}$$

where δ_{jj_k} is the Kronecker delta. The essential and natural boundary conditions are obtained and listed in the following two columns. In analysis, only one condition, either essential or natural, needs to be specified in each group.

Essential boundary conditions	Natural boundary conditions	
w	$\left[Q_x + \bar{\eta}_x - \bar{\lambda}_{x,x} - \frac{1}{2}\bar{\lambda}_{xy,y} + \sum_{k=1}^{n_1}\mu_{jk}(Q_{55}^{jk+1})^2\left(\tilde{S}_{jk} + \frac{\partial w}{\partial x}\right)\right]n_x + \left[Q_y + \bar{\eta}_y - \bar{\lambda}_{y,y} - \frac{1}{2}\bar{\lambda}_{xy,x} + \sum_{k=1}^{n_1}\nu_{jk}(Q_{44}^{jk+1})^2\left(\tilde{T}_{jk} + \frac{\partial w}{\partial y}\right)\right]n_y$	
$\frac{\partial w}{\partial x}$	$\bar{\lambda}_x n_x + \frac{1}{2}\bar{\lambda}_{xy}n_y$	
$\frac{\partial w}{\partial y}$	$\frac{1}{2}\bar{\lambda}_{xy}n_x + \bar{\lambda}_y n_y$	
U_j	$N^j_x n_x + N^j_{xy}n_y$	$j = 0, 1, \ldots, n$
V_j	$N^j_{xy}n_x + N^j_y n_y$	$j = 0, 1, \ldots, n$
$\tilde{S}_j$	$M^j_x n_x + M^j_{1xy}n_y$	$j = 1, 2, \ldots, n-1$
$\tilde{T}_j$	$M^j_{2xy}n_x + M^j_y n_y$	$j = 1, 2, \ldots, n-1$

(20)

The resultant forces and moments in Eqs. (19) and (20) are defined as follows,

$$(Q_x, Q_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz})\, dz$$

$$(N^j_x, N^j_y, N^j_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy})\Phi^j\, dz$$

$$(M^j_x, M^j_y, M^j_{1xy}, M^j_{2xy}) = \int_{-h/2}^{h/2} (\sigma_x\bar{\Psi}_1^j, \sigma_y\bar{\Psi}_2^j, \tau_{xy}\bar{\Psi}_1^j, \tau_{xy}\bar{\Psi}_2^j)\, dz$$

$$(\bar{\lambda}_x, \bar{\lambda}_y, \bar{\lambda}_{xy}) = \int_{-h/2}^{h/2} (\sigma_x\lambda_1, \sigma_y\lambda_2, \tau_{xy}\lambda_1 + \tau_{xy}\lambda_2)\, dz \tag{21}$$

$$(\bar{\eta}_x, \bar{\eta}_y) = \int_{-h/2}^{h/2}\left(\tau_{xz}\frac{d\lambda_1}{dz}, \tau_{yz}\frac{d\lambda_2}{dz}\right) dz$$

$$(Q^j_x, Q^j_y) = \int_{-h/2}^{h/2}\left(\tau_{xz}\frac{d\Phi^j}{dz}, \tau_{yz}\frac{d\Phi^j}{dz}\right) dz$$

$$(R^j_x, R^j_y) = \int_{-h/2}^{h/2}\left(\tau_{xz}\frac{d\bar{\Psi}_1^j}{dz}, \tau_{yz}\frac{d\bar{\Psi}_2^j}{dz}\right) dz$$

Closed-Form Solution

In order to verify that the present theory can be used for composite laminates with nonrigidly bonded interfaces, the cylindrical bending of an infinitely long strip examined by Pagano[1] is studied. Figure 3 is the configuration of the composite laminate. The displacement field is reduced to be functions of x and z only. In other words, all of the derivatives with respect to y in Eqs. (17), (19), and (20) vanish. By combining Eqs. (21) with Eqs. (19) along with the stresses and strains from Eqs. (3) and (17), the final governing equations can be expressed in terms of independent variables,

$$(A_{55} + \bar{A}_{55} + \gamma_1 + \bar{\gamma}_1)w_{,xx} + \sum_{j=0}^{n}[(B^j_{55} + \bar{B}^j_{55})U_{j,x} - \bar{B}^j_{11}U_{j,xxx}] + \sum_{m=1}^{n-1}[(C^m_{55} + \bar{C}^m_{55})\tilde{S}_{m,x} - \bar{C}^m_{11}\tilde{S}_{m,xxx}] + \sum_{k=1}^{n_1}\mu_{jk}(Q_{55}^{jk+1})^2\left(\tilde{S}_{jk,x} + \frac{\partial^2 w}{\partial x^2}\right) - \bar{\gamma}_1 w_{,xxxx} + P_z = 0$$

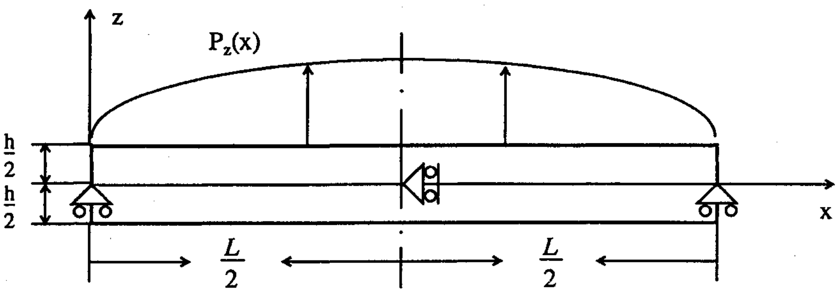


Fig. 3 Cylindrical bending of an orthotropic laminate: $E_1 = 25 \times 10^6$ psi; $E_2 = E_3 = 1 \times 10^6$ psi; $G_{12} = G_{13} = 0.5 \times 10^6$ psi; $G_{23} = 0.2 \times 10^6$ psi; $v_{12} = v_{13} = v_{23} = 0.25$.

$$\sum_{j=0}^{n} (D_{11}^{lj} U_{j,xx} - D_{55}^{lj} U_j) + \sum_{m=1}^{n-1} (E_{11}^{lm} \tilde{S}_{j,xx} - E_{55}^{lm} \tilde{S}_j) + \bar{B}_{11}^{l} w_{,xxx} - (B_{55}^{l} + \bar{B}_{55}^{l}) w_{,x} = 0 \tag{22}$$

$$l = 0, 1, \ldots, n$$

$$\sum_{j=0}^{n} (E_{11}^{jt} U_{j,xx} - E_{55}^{jt} U_j) + \sum_{m=1}^{n-1} (F_{11}^{tm} \tilde{S}_{j,xx} - F_{55}^{tm} \tilde{S}_j) + \bar{C}_{11}^{t} w_{,xxx} - (C_{55}^{t} + \bar{C}_{55}^{t}) w_{,x} - \mu_{j_k} (Q_{55}^{j_k+1})^2 \left(\tilde{S}_{j_k} + \frac{\partial w}{\partial x} \right) \delta_{j_k t} = 0$$

$$t = 1, 2, \ldots, n-1; \qquad j_k \in \Pi$$

where $\bar{B}_{11}^{j}$, B_{55}^{j}, $\bar{B}_{55}^{j}$, $\bar{C}_{11}^{m}$, C_{55}^{m}, $\bar{C}_{55}^{m}$, D_{11}^{lj}, D_{55}^{lj}, E_{11}^{lm}, E_{55}^{lm}, F_{11}^{tm}, F_{55}^{tm}, A_{55}, $\bar{A}_{55}$, γ_2, $\bar{\gamma}_1$, and $\bar{\gamma}_2$ are the coefficients of laminate properties. They can be expressed as follows,

$$A_{55} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \, dz, \quad B_{55}^{j} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \Phi^{j}_{,z} \, dz$$

$$C_{55}^{m} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \Psi_{1,z}^{m} \, dz, \quad \gamma_2 = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \lambda_{1,z} \, dz$$

$$\bar{\gamma}_1 = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{11}^{(i)} \lambda_{1,z}^{2} \, dz, \quad \bar{\gamma}_2 = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \lambda_{1,z}^{2} \, dz$$

$$\bar{B}_{11}^{j} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{11}^{(i)} \Phi^{j} \lambda_1 \, dz, \quad \bar{C}_{11}^{m} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{11}^{(i)} \Psi_{1}^{m} \lambda_1 \, dz$$

$$\bar{A}_{55} = \gamma_2 \tag{23}$$

$$\bar{B}_{11}^{j} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \Phi^{j}_{,z} \lambda_{1,z} \, dz, \quad \bar{C}_{55}^{m} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \Psi_{1,z}^{m} \lambda_{1,z} \, dz$$

$$D_{11}^{lj} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{11}^{(i)} \Phi^{l} \Phi^{j} \, dz, \quad E_{11}^{lm} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{11}^{(i)} \Phi^{l} \Psi_{1}^{m} \, dz$$

$$D_{55}^{lj} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \Phi^{l}_{,z} \Phi^{j}_{,z} \, dz, \quad E_{55}^{lm} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \Phi^{l}_{,z} \Psi_{1,z}^{m} \, dz$$

$$F_{11}^{tm} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{11}^{(i)} \Psi_{1}^{t} \Psi_{1}^{m} \, dz, \quad F_{55}^{tm} = \sum_{j=1}^{n} \int_{z_{i-1}}^{z_i} Q_{55}^{(i)} \Psi_{1,z}^{t} \Psi_{1,z}^{m} \, dz$$

Since the laminate of interest is simply supported at $x = 0$ and L, the boundary conditions are as follows,

$$w(0) = w(L) = 0, \quad \bar{\lambda}_x(0) = \bar{\lambda}_x(L) = 0$$

$$N_x^m(0) = N_x^m(L) = 0 \quad m = 0, 1, 2, \ldots, n \tag{24}$$

$$M_x^m(0) = M_x^m(L) = 0 \quad m = 1, 2, \ldots, n-1$$

The loading P_z is assumed to be sinusoidal distribution, i.e.,

$$P_z = P_0 \sin(\beta x) \tag{25}$$

where $\beta = \pi/L$. In order to satisfy the boundary conditions listed in Eqs. (24), the following displacements are assumed,

$$u = \underline{u} \cos(\beta x) \qquad w = \underline{w} \sin(\beta x)$$

$$U_j = \underline{U}_j \cos(\beta x) \qquad j = 0, 1, 2, \ldots, n \tag{26}$$

$$\tilde{S}_j = \underline{S}_j \cos(\beta x) \qquad j = 1, 2, \ldots, n-1$$

where $\underline{u}$, $\underline{w}$, $\underline{U}_j$, and $\underline{S}_j$ are coefficients to be determined. It is obvious that w satisfies the boundary conditions, i.e., $w(0) = w(L) = 0$. The satisfaction of the remaining boundary conditions can also be verified. By expressing Eqs. (21) in terms of displacements, it may have

$$\bar{\lambda}_x = \sum_{j=0}^{n} \bar{B}_{11}^{j} U_{1,x} + \sum_{j=1}^{n-1} \bar{C}_{11}^{j} \tilde{S}_{j,x} + \bar{\gamma}_1 w_{,xx}$$

$$N_x^m = \sum_{j=0}^{n} D_{11}^{mj} U_{1,x} + \sum_{j=1}^{n-1} E_{11}^{mj} \tilde{S}_{m,x} \bar{B}_{11}^{m} w_{,xx} \tag{27}$$

$$M_x^m = \sum_{j=0}^{n} E_{11}^{mj} U_{1,x} + \sum_{j=1}^{n-1} F_{11}^{mj} \tilde{S}_{m,x} + \bar{C}_{11}^{m} w_{,xx}$$

It is clear that the resultants in Eqs. (27) are of sine functions. They all satisfy the boundary conditions listed in Eqs. (24). Therefore, Eqs. (26) can be a set of solutions to the governing equations. By substituting Eqs. (26) into Eqs. (22), the coefficients $\underline{w}$, $\underline{U}_j$, and $\underline{S}_j$ can be determined and a closed-form solution can be obtained.

Embedded-Layer Approach

Another method to study composite laminates with a nonrigidly bonded interface is to introduce an embedded layer. The controversy of this method is the determination of the thickness of the embedded layer and the material properties for the layer. In this study, the thickness of the embedded layer a is assumed to be 0.0764 mm.[36] Since only the shear slip is considered, it is possible to simulate the nonrigid bonding by adjusting the transverse shear modulus G_{13}. The finite element method presented in Ref. 27 is used to study the deformation of the composite laminates with embedded layers. The numerical results are shown in a following section.

Although the slip analysis and the embedded technique are derived from different bases, a relation between them can be

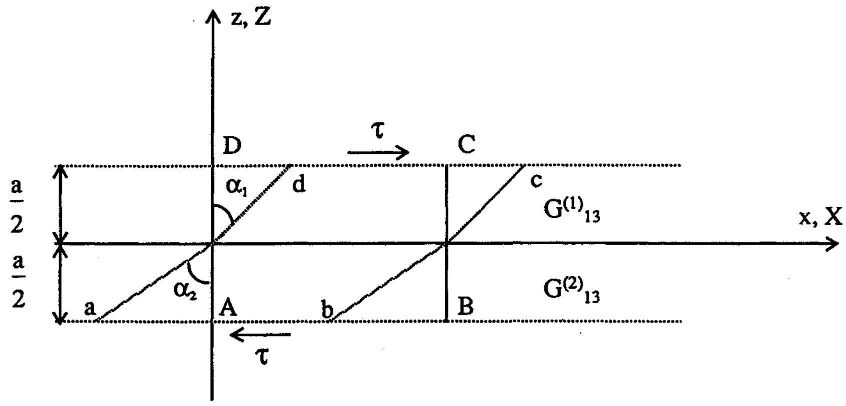
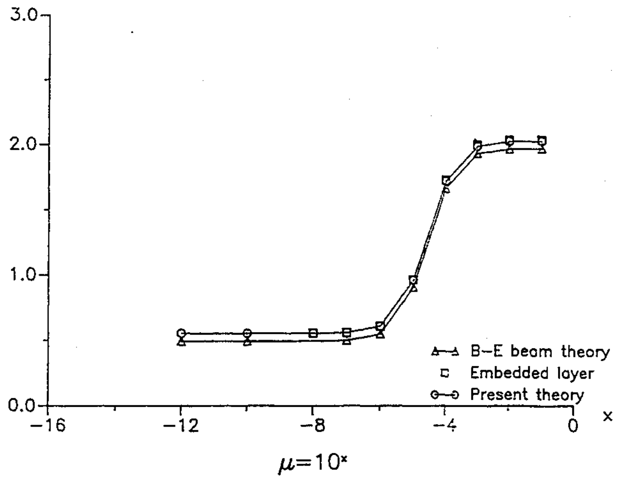
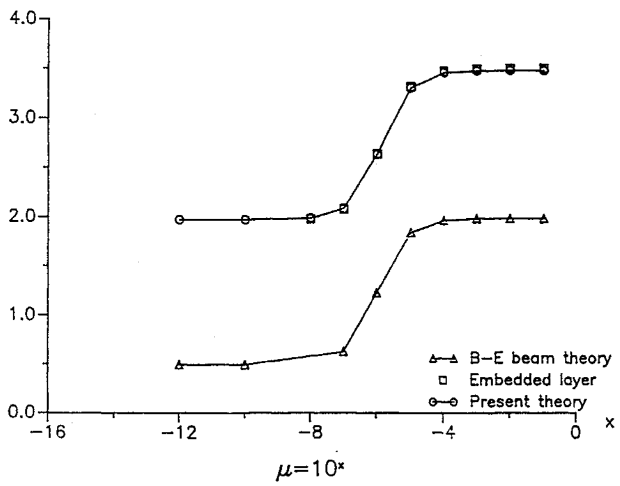


Fig. 4 Pure shear deformation in embedded layer.

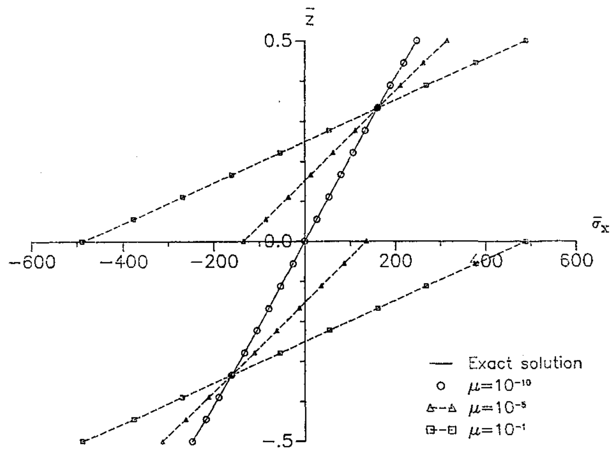


a) $S = 20$

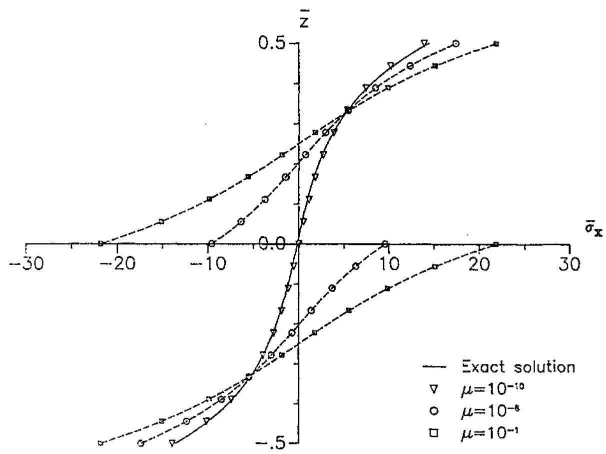


b) $S = 4$

Fig. 5 Maximum deflection of [0/0] laminates as a function of bonding coefficient.



a) $S = 20$



b) $S = 4$

Fig. 6 In-plane normal stress through the thickness for [0/0] laminates.

found from the following analysis. Assuming that an embedded thin layer will undergo a pure shear deformation, as shown in Fig. 4, i.e., *ABCD* becomes *abcd* after deformation due to a shear stress τ acting on the layer, the displacement field can be written as follows,

$$x = X + \tan(\alpha_i)Z \qquad i = 1 \text{ if } Z > 0; \quad i = 2 \text{ if } Z < 0$$

$$z = Z \tag{28}$$

Thus, τ can be expressed as

$$\tau = G_{13}^{(1)} \tan(\alpha_1) = G_{13}^{(2)} \tan(\alpha_2) \tag{29}$$

In addition, the difference of the displacements at $z = \pm a/2$ is of

$$u\left(x, \frac{a}{2}\right) - u\left(x, -\frac{a}{2}\right) = [\tan(\alpha_1) + \tan(\alpha_2)]\frac{a}{2} \tag{30}$$

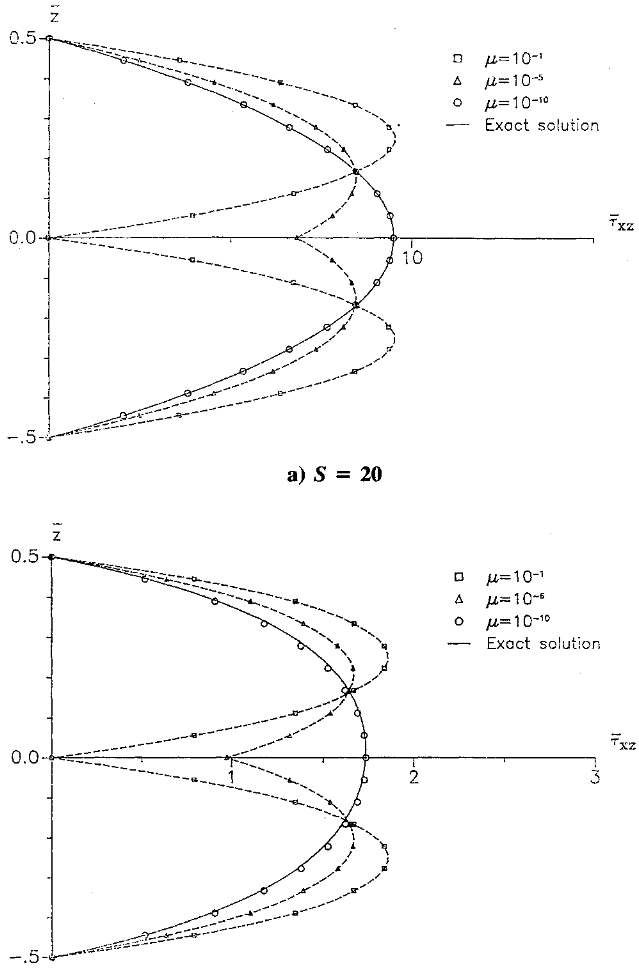


Fig. 7 Transverse shear stress through the thickness for [0/0] laminates.

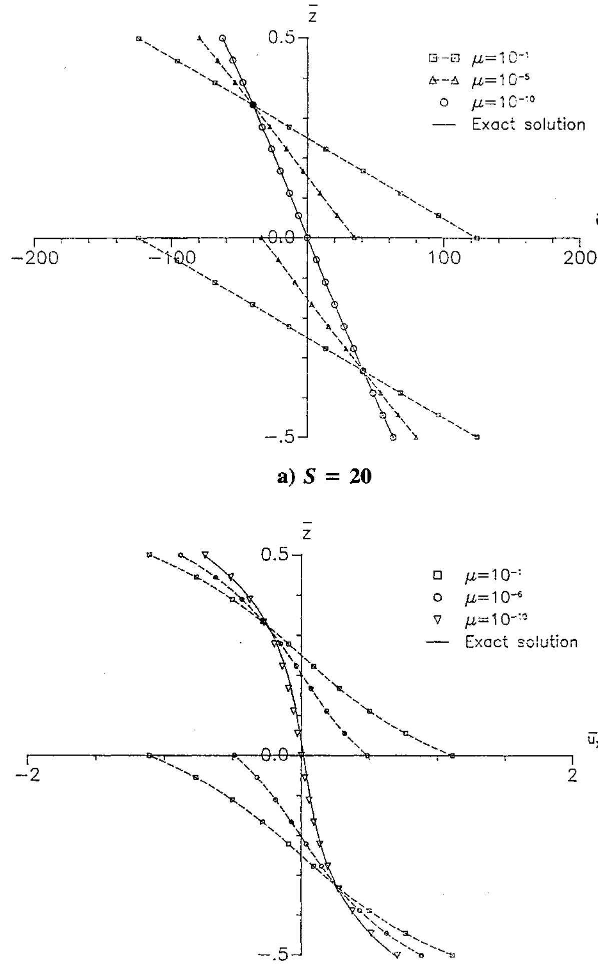


Fig. 8 In-plane displacement through the thickness for [0/0] laminates.

If the embedded layer is thin enough to simulate the real interface, the difference described in Eq. (30) can be regarded the same as the interlayer shear slip defined in Eqs. (5). Accordingly, the relations between the embedded-layer approach and the ISST under the linear shear slip law, Eqs. (12), is

$$\mu = \frac{a}{2}\frac{G_{13}^{(2)} + G_{13}^{(1)}}{G_{13}^{(1)}G_{13}^{(2)}} \tag{31}$$

Results and Discussions

Case 1: [0/0] Laminates with a Nonrigid Interface on the Middle Plane

[0/0] composite laminates with a nonrigid interface on the middle plane are studied. The numerical results from the closed-form solution are summarized in the following with use of nondimensional terms defined by Pagano,[1] i.e.,

$$\bar{\sigma}_x = \frac{\sigma_x[(L/2), z]}{P_o}, \qquad \bar{\tau}_{xy} = \frac{\tau_{xy}(0, z)}{P_o}, \qquad \bar{u} = \frac{E_3 u(0, z)}{hP_o}$$

$$\bar{w} = \frac{100h^3E_3w(L/2)}{L^4P_o}, \qquad \bar{z} = \frac{z}{h} \tag{32}$$

Transverse Shear Effect

The maximum deflections calculated from different approaches are presented in Fig. 5 as a function of interfacial shear bonding coefficient μ for both $S = 4$ and 20. S is the ratio of the length to the thickness of the composite laminate, i.e., L/h. It is usually called aspect ratio[1] and is an index of laminate thickness. Small S is for thick laminates, whereas large S is for thin. The approach of the Bernoulli-Euler beam theory can be referred to in Ref. 29. It is concluded that the results from ISST are quite close to those from the Bernoulli-Euler beam theory when $S = 20$. However, the difference between them is very considerable for $S = 4$. This difference is apparently due to the transverse shear effect. In addition, the results from the embedded-layer technique are very close to those from ISST for both $S = 4$ and 20. It verifies the feasibility of using Eq. (31) for composite laminates with a nonrigid interface.

Bonding Coefficient

As shown in Fig. 5, the maximum deflection reaches an upper limit as μ approaches 1 while lower limit as μ goes to zero. The former corresponds to a rigidly bonded interface, whereas the latter a completely debonded interface. By setting $\mu = 10^{-10}$ for rigid bonding and $\mu = 10^{-1}$ for completely debonded interface, it is verified that the deflections from ISST are the same as those presented in Ref. 26. In addition, it is concluded from Fig. 5 that the deflection changes dramatically within a small range of μ. This implies that nonrigid bonding can have a significant effect on the deflection when the interfacial shear bonding approaches the critical range. However, beyond the critical range, the composite laminate will quickly lose its integrity and behave like two independent layers with a lubricated interface.[33] This phenomenon has also been re-

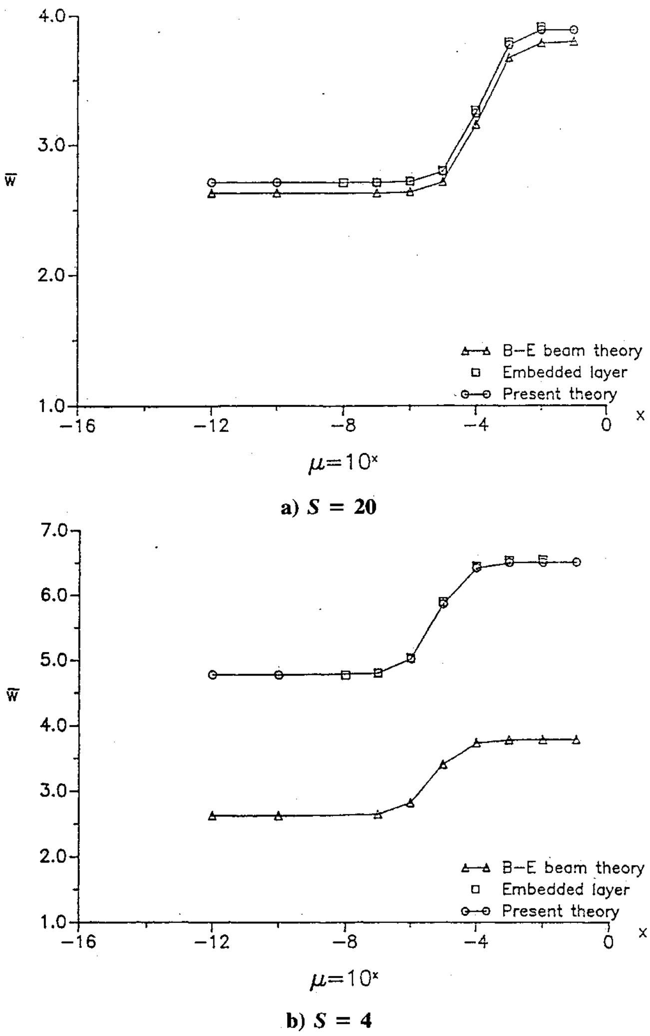


Fig. 9 **Maximum deflection of [90/0] laminates as a function of bonding coefficient.**

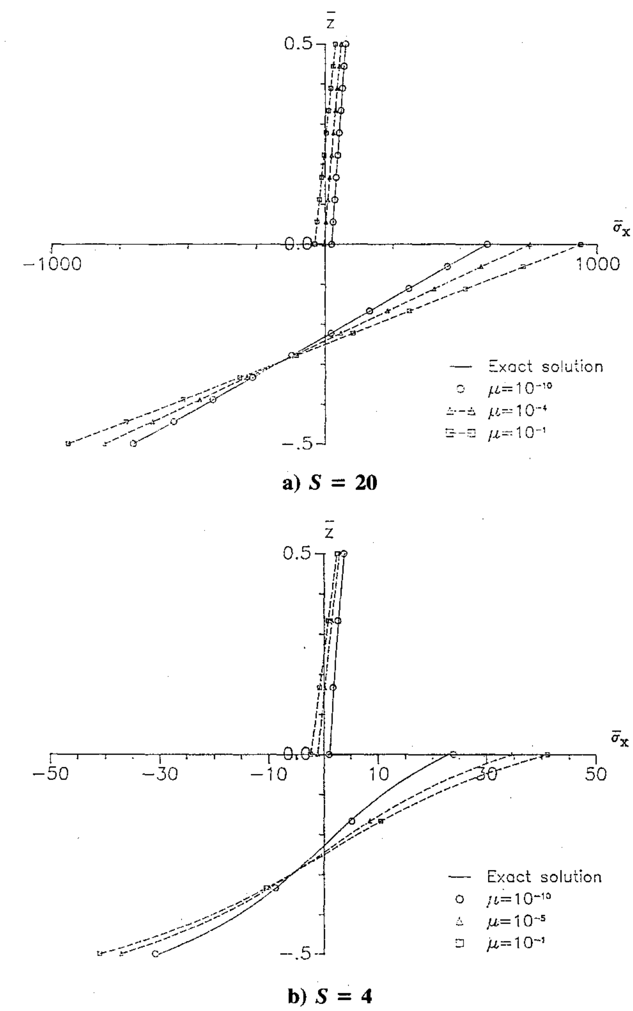


Fig. 10 **In-plane normal stress through the thickness for [90/0] laminates.**

ported in Ref. 33 though the results beyond the critical range are not presented in detail. In addition, it needs to be pointed out that the critical range seems to change as the material property or geometry change.

Maximum Stresses

Figures 6–8 show the distributions of in-plane stress, transverse shear stress, and in-plane displacement as functions of interfacial shear bonding coefficient μ for S = 4 and 20. The solid lines represent the results from Ref. 1 for rigidly bonded laminates, and the dashed lines represent the results from ISST for different shear bonding coefficients. By setting $\mu = 10^{-10}$, the results from ISST are very close to those from Ref. 1.

From Figs. 6–8, it is also concluded that the in-plane normal stress bcomes larger and larger as the bonding gets less and less rigid. For $\mu = 10^{-5}$ and 10^{-1}, the maximum normal stresses increase 27 and 97% in S = 20 laminates, respectively, whereas they increase 25 and 56% in S = 4 laminates, respectively. In addition, the maximum transverse shear stress for S = 20 is always smaller than that of rigid bonding. However, the maximum value for S = 4 with nonrigid bonding can exceed the value of rigid bonding. This is other evidence to indicate that the transverse shear effect is very important in thick laminate analysis. Besides, these results seem to further support the conclusion in Ref. 33 that the damage in a laminate with nonrigid bonding can take place even before the prediction from the theory based on the rigid bonding assumption.

In addition, it is shown in Fig. 7 that the delaminated interface has no ability to carry any shear stress, although the nonrigid interface, i.e., $\mu = 10^{-5}$, still can transfer shear stress up to some extent.

Singular Points

It is interesting to point out that there are four singular points in which the normal stress, interlaminar shear stress, and in-plane displacement remain constants even though the bonding condition changes. The locations of the singular points are near $z = \pm h/3$ for both in-plane normal stress and displacement in S = 4 and 20 laminates, whereas they are around $z = \pm h/6$ for transverse shear stress. It is believed that these singular points have special interest for composite laminates with embedded sensors.

Case 2: [90/0] Laminates with a Nonrigidly Bonded Interface on the Middle Plane

The numerical results for [90/0] laminates with a nonrigidly bonded interface on the middle plane are shown in Figs. 9–12. For maximum deflections, similar conclusions as obtained in case 1 can be drawn. By comparing the results from the present theory and the Bernoulli-Euler beam approach, it is concluded again that the shear effect is very important in the study of composite laminates with a nonrigidly bonded interface, especially in thick laminate analysis. Besides, the results of transverse deflection, in-plane stress and displacement, and out-of-plane stress from ISST for both S = 4 and 20 are the same as the those given in Ref. 26 when μ = 0. This implies that the ISST also can be used for both thin and thick laminates with rigidly bonded interface.

For both S = 4 and 20 [90/0] laminates, the maximum values of in-plane normal stress and interlaminar shear stress in-

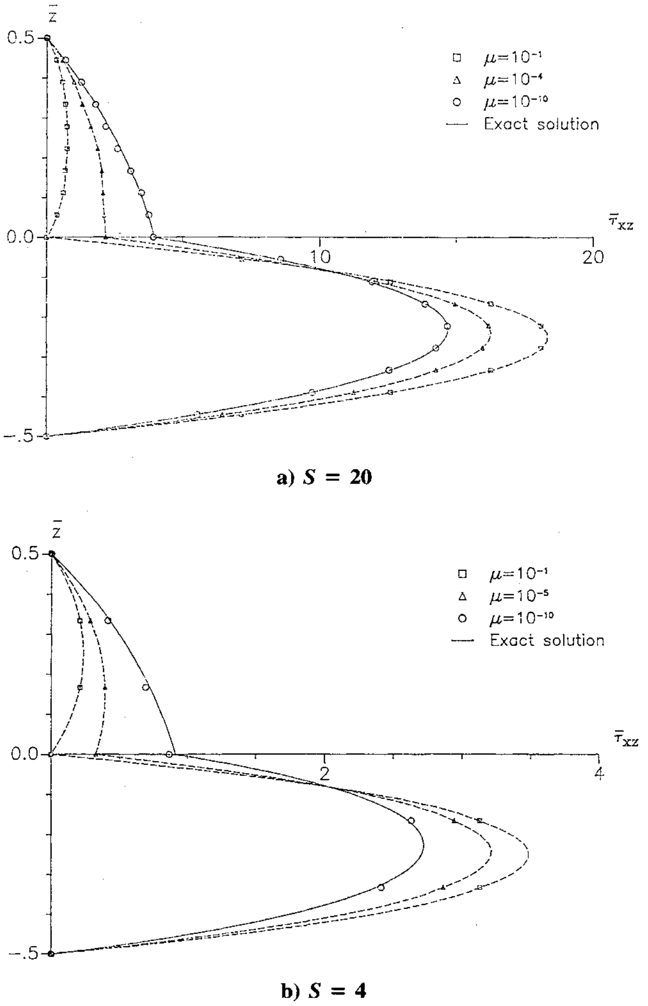


Fig. 11 Transverse shear stress through the thickness for [90/0] laminates.

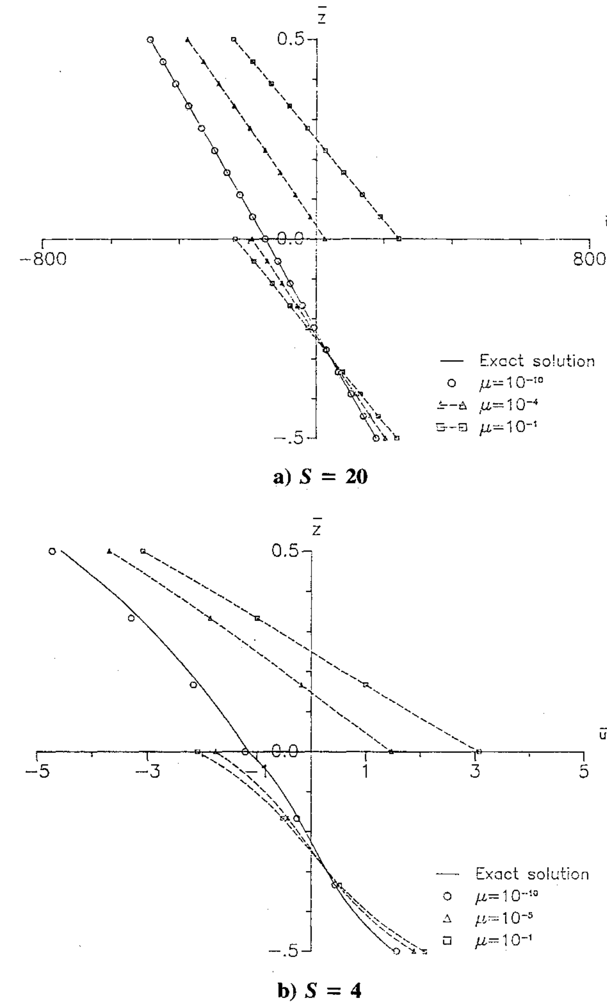


Fig. 12 In-plane displacement through the thickness for [90/0] laminates.

crease as the bonding gets less and less rigid. For $\mu = 10^{-4}$ and 10^{-1}, the increases of the maximum normal stress in S = 20 laminate are about 15 and 34%, respectively, whereas the increases of the maximum shear stress are around 11 and 24%, respectively. In addition, it can be found that the resultant shear force is independent of bonding coefficient μ. However, there are only two singular points that are located in the 0 layer. For both in-plane normal stress and displacement, it is around $z = -0.29h$, whereas it is near $z = -0.21h$ for transverse shear stress.

Conclusions

Excellent results have been obtained from the interlayer shear slip theory for transverse deflection, in-plane displacement, and in-plane normal stress in both symmetrical and unsymmetrical laminates. Because of the consideration of interlaminar shear stress continuity conditions, the interlaminar shear stress from the present theory can be calculated accurately from the constitutive equations directly. Numerical results have concluded that the interlayer shear slip theory can be used to study both thin and thick composite laminates with rigid and nonrigid interfaces. In addition, the present theory can be extended to finite element formulation and used to study both cross-ply and angle-ply laminates with nonrigid bonding at various boundary conditions.

References

[1]Pagano, N. J., "Exact Solutions for Composite Laminates in Cylindrical Bending," *Journal of Composite Materials*, Vol. 3, July 1969, pp. 398–411.

[2]Pagano, N. J., "Exact Solutions for Rectangular Bidirectional Composites and Sandwich Plates," *Journal of Composite Materials*, Vol. 4, Jan. 1970, pp. 20–34.

[3]Whitney, J. M., "Stress Analysis of Thick Laminated Composite and Sandwich Plates," *Journal of Composite Materials*, Vol. 6, Oct. 1972, pp. 426–440.

[4]Whitney, J. M., and Pagano, N. J., "Shear Deformation in Heterogeneous Anisotropic Plates," *Journal of Applied Mechanics*, Vol. 37, No. 4, Dec. 1970, pp. 1031–1036.

[5]Noor, A. K., and Burton, W. S., "Assessment of Shear Deformation Theories for Multilayered Composite Plates," *Applied Mechanics Review*, Vol. 42, No. 1, Jan. 1989, pp. 1–13.

[6]Kapania, R. K., and Raciti, S., "Recent Advances in Analysis of Laminated Beams and Plates, Part I: Shear Effects and Buckling," *AIAA Journal*, Vol. 27, No. 7, 1987, pp. 923–934.

[7]Reddy, J. N., "A Review of Refined Theories of Laminated Composite Plates," *Shock and Vibration Digest*, Vol. 22, No. 1, 1990, pp. 3–17.

[8]Yang, P. C., Norris, C. H., and Stavsky, Y., "Elastic Wave Propagation in Heterogeneous Plates," *International Journal of Solids and Structures*, Vol. 2, No. 4, 1966, pp. 665–684.

[9]Nelson, R. B., and Lorch, D. R., "A Refined Theory of Laminated Orthotropic Plates," *Journal of Applied Mechanics*, Vol. 41, No. 1, March 1974, pp. 177–183.

[10]Reissner, E., "On Transverse Bending of Plate, Including the Effect of Transverse Shear Deformation," *International Journal of Solids and Structures*, Vol. 11, No. 5, 1975, pp. 569–573.

[11]Lo, K. H., Christensen, R. M., and Wu, E. M., "A High-Order Theory of Plate Deformation Part 2: Laminated Plates," *Journal of Applied Mechanics*, Vol. 44, No. 4, Dec. 1977, pp. 669–676.

[12]Reddy, J. N., "A Simple Higher-Order Theory for Laminated Composites Plates," *Journal of Applied Mechanics*, Vol. 51, No. 4, Dec. 1984, pp. 745–752.

[13]Reddy, J. N., "A Generalization of Two-Dimensional Theories of Laminated Composite Plates," *Communications in Applied Numerical Methods*, Vol. 3, No. 1, 1987, pp. 173–180.

[14]Pagano, N. J., and Pipes, R. B., "Some Observations on the Interlaminar Strength of Composite Laminates," *International Journal of Mechanical Sciences*, Vol. 15, No. 8, 1973, pp. 679–688.

[15]Liu, D., "Impact-Induced Delamination—A Veiw of Bending Stiffness Mismatching," *Journal of Composite Materials*, Vol. 22, July 1988, pp. 674–692.

[16]Lo, K. H., Christensen, R. M., and Wu, E. M., "Stress Solution Determination for High Order Plate Theory," *International Journal of Solids and Structures*, Vol. 14, No. 8, 1978, pp. 655–662.

[17]Ambartsumyan, S. A., *Theory of Anisotropic Plates*, edited by J. E. Ashton, Technomic, Stamford, CT, 1970.

[18]Librescu, L., and Reddy, J. N., "A Critical Evaluation on Generalization of the Theory of Anisotropic Laminated Composite Panels," *Proceedings of the American Society for Composites*, First Technical Conf., Lancaster, Inc., 1986, Lancaster, PA, pp. 471–489.

[19]Mau, S. T., Tong, P., and Pian, T. H. H., "Finite Element Solutions for Laminated Thick Plates," *Journal of Composite Materials*, Vol. 6, April 1972, pp. 304–311.

[20]Spilker, R. L., "A Hybrid-Stress Finite-Element Formulation for Thick Multilayer Laminates," *Computers and Structures*, Vol. 26, No. 6, 1980, pp. 507–514.

[21]Pagano, N. J., "Stress Fields in Composite Laminates," *International Journal of Solids and Structures*, Vol. 14, No. 5, 1978, pp. 385–400.

[22]DiSciuva, M., "Development of An Anisotropic Multilayered Shear-Deformable Rectangular Plate Element," *Computers and Structures*, Vol. 21, No. 4, 1985, pp. 789–796.

[23]Toledano, A., and Murakami, H., "A Composite Plate Theory for Arbitrary Laminate Configurations," *Journal of Applied Mechanics*, Vol. 54, No. 1, March 1987, pp. 181–189.

[24]Reissner, E., "On a Certain Mixed Variational Principle and a Proposed Application," *International Journal for Numerical Methods in Engineering*, Vol. 20, No. 7, July 1984, pp. 1366–1368.

[25]Hinrichsen, R. L., and Palazotto, A. N., "Nonlinear Finite Element Analysis of Thick Composite Plates Using Spline Functions," *AIAA Journal*, Vol. 24, No. 11, 1986, pp. 1836–1842.

[26]Lu, X., and Liu, D., "An Interlaminar Shear Stress Continuity Theory for both Thin and Thick Composite Laminates," *Journal of Applied Mechanics* (to be published).

[27]Lee, C. Y., Liu, D., and Lu, X., "Static and Vibration Analysis of Laminated Beams by Using an Interlaminar Shear Stress Continuity Theory," *International Journal for Numerical Methods in Engineering* (to be published).

[28]Newmark, N. M., Seiss, C. P., and Viest, I. M., "Tests and Analysis of Composite Beams with Incomplete Interaction," *Proceedings of the Society for Experimental Stress Analysis*, Brookfield Center, Vol. 9, No. 1, Brookfield, CT, 1951, pp. 73–79.

[29]Goodman, J. R., and Popkov, E. P., "Layered Beam Systems with Interlayer Slip," *Journal of Structural Division*, ASCE, Vol. 24, No. ST11, Nov. 1968, pp. 2535–2547.

[30]Goodman, J. R., "Layered Wood Systems with Interlayer Slip," *Wood Science*, Vol. 1, No. 3, 1969, pp. 148–158.

[31]Thompson, E. G., Goodman, J. R., and Vanderbilt, M. D., "Finite Element Analysis of Layered Wood Systems," *Journal of Structural Division*, ASCE, No. ST12, Vol. 101, Dec. 1975, pp. 2659–2672.

[32]Vanderbilt, M. D., Goodman, J. R., and Criswell, W. E., "Service and Overload Behavior of Wood Joist Floor Systems," *Journal of Structural Division*, ASCE, Vol. 100, No. ST1, Jan. 1974, pp. 11–29.

[33]Toledano, A., and Murakami, H., "Shear-deformable Two-Layer Plate theory with Interlayer Slip," *Journal of Engineering Mechanics Division*, ASCE, Vol. 114, No. 3, April 1988, pp. 605–623.

[34]Rao, K. M., and Ghosh, B. G., "Imperfectly Bonded Unsymmetric Laminated Beam," *Journal of Engineering Mechanics Division*, ASCE, Vol. 106, No. EM4, Aug. 1980, pp. 685–697.

[35]Fazio, P., Hussein, R., and Ha, K. H., "Beam-Columns with Interlayer Slips," *Journal of Engineering Mechanics Division*, ASCE, Vol. 108, No. EM2, April 1982, pp. 354–366.

[36]Vinson, J. R., and Sierakowski, R. L., *The Behavior of Structures Composed of Composite Materials*, Martinus Nijhoff, Dordrecht, The Netherlands, 1986.

[37]Pagano, N. J., and Soni, S. R., *Interlaminar Response of Composite Materials*, edited by N. J. Pagano, Elsevier, NY, 1989, pp. 1–68.

[38]Lee, C. Y., and Liu, D., "An Interlaminar Stresses Continuous Theory for Laminated Composite Beam Analysis," *AIAA Journal*, Vol. 29, No. 11, 1991, pp. 2010-2012.